Quantum values of cooperative quantum games

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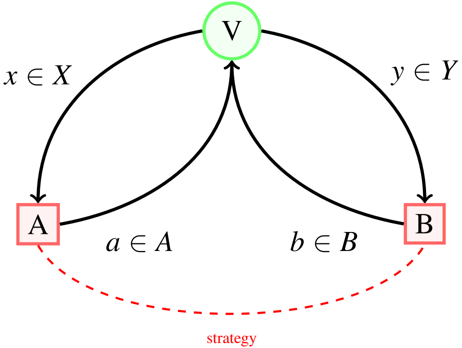
MOTIVATION

- Non-local games have been important in the process of understanding the entanglement.
- It has been clear from recent results that the theory of non-local games provides also a way to construct interesting examples of (operator) algebras.
- Recently non-local games have been used to show that the Connes Embedding Problem has a negative answer.
- The value of a nonlocal game, which is the supremum of the probability of winning the game over all allowed strategies, has been an important ingredient for the proof of CEP and the equivalent Tsirelson conjecture about separation of different mathematical models for entanglement;
- It seems that non-local games with quantum inputs/outputs suggest further advantages. Such games and quantum game values arising from different strategies and their connection with different operator space tensor norms will be the main focus in my talk.

Classical non-local game

A non-local game is a tuple $\mathcal{G} = (X, Y, A, B, \lambda, \pi)$, where X, Y, A and B are finite sets, $\lambda : X \times Y \times A \times B \to \{0, 1\}$, π is a probability distribution on $X \times Y$.

with prob. $\pi(x, y)$



$\lambda(x, y, a, b) = 0 \rightsquigarrow$ loose; $\lambda(x, y, a, b) = 1 \rightsquigarrow$ win

NON-LOCAL GAMES: STRATEGIES

A deterministic strategy: $f : X \to A, \; g : Y \to B$.

It is a perfect (or winning) strategy if $\lambda(x, y, f(x), g(y)) = 1$ for all $x \in X$, $y \in Y$

More generally: The players possess a set $\{(f_i, g_i)\}_{i=1}^{k}$ of deterministic strategies and employ randomness to choose which one to apply, according to a probability distribution $(\lambda_i)_{i=1}^{k}$.

> Non-deterministic strategies: to the same input (x, y) the players may reply in different rounds with different outputs (a, b) and (a', b').

$\rightsquigarrow$ A probabilistic strategy: $\{(p(a, b|x, y))_{(a,b)\in A\times B} : (x, y) \in X \times Y\}$, where $p(\cdot, \cdot|x, y)$: a probability distribution for each (x, y).

$p(a, b|x, y)$ is the probability of reply (a, b) to the question (x, y).

p is a perfect strategy if $\lambda(x, y, a, b) = 0 \implies p(a, b|x, y) = 0$.

NO-SIGNALLING STRATEGIES

A probabilistic strategy: $\{(p(a,b|x,y))_{(a,b)\in A\times B} : (x,y)\in X\times Y\}$, where $p(\cdot,\cdot|x,y)$: a probability distribution for each (x,y).

$p=(p(a,b|x,y))$ no-signalling if $\exists$ well-defined marginals:

$$p(a|x)=\sum_{b\in B}p(a,b|x,y'),\ p(b|y)=\sum_{a\in A}p(a,b|x',y)\ \text{Notation: } \mathcal{C}_{\mathrm{ns}}.$$

A correlation p is called

- local if it is a convex combination of product correlations $p_1(a|x)p_2(b|y)$ Notation: $\mathcal{C}_{\mathrm{loc}}$.
- quantum if
$$p(a,b|x,y)=\langle(E_{x,a}\otimes F_{y,b})\xi,\xi\rangle,$$
where $(E_{x,a})_{a\in A}$ $(F_{y,b})_{b\in B}$ fin. dim. POVM's (i.e. $E_{x,a}\geq 0$, $\sum_a E_{x,a}=1$) Notation: $\mathcal{C}_{\mathrm{q}}$.
- quantum commuting if
$$p(a,b|x,y):=\langle E_{x,a}F_{y,b}\xi,\xi\rangle,$$
where $(E_{x,a})_{a\in A}$ and $(F_{y,b})_{b\in B}$ POVM's with $E_{x,a}F_{y,b}=F_{y,b}E_{x,a}$ Notation: $\mathcal{C}_{\mathrm{qc}}$.

$$\mathcal{C}_{\mathrm{loc}}\subsetneq\mathcal{C}_{\mathrm{q}}\subsetneq\overline{\mathcal{C}_{\mathrm{q}}}\subsetneq\mathcal{C}_{\mathrm{qc}}\subsetneq\mathcal{C}_{\mathrm{ns}}$$

VALUE OF A GAME

p is a perfect strategy if $\lambda(x,y,a,b)=0 \implies p(a,b|x,y)=0$.
If no perfect strategy in $\mathcal{C}_t$ one asks about the $\mathcal{C}_t$-winning rate (t-value):

$$\omega_t(\mathcal{G},\pi)=\sup_{p\in\mathcal{C}_t}\sum_{x,y,a,b}\pi(x,y)p(a,b|x,y)\lambda(x,y,a,b).$$

- $\omega_t(\mathcal{G},\pi)=1$ iff $\exists$ perfect strategy $p\in\mathcal{C}_t$;
- CHSH -game: $X=Y=A=B=\{0,1\}$, $\lambda(x,y,a,b)=1 \Leftrightarrow xy=a+b$, π-uniform. Then $\omega_{loc}(\mathcal{G})=0,75$, but $\omega_q(\mathcal{G},\pi)=\omega_{qc}(\mathcal{G},\pi)\approx 0,85$.
- by an extreme point argument, $\omega_{loc}(\mathcal{G})$ is obtained by deterministic strategy;
- $\left.\begin{array}{l}\omega_q(\mathcal{G})=\sup\sum_{x,y,a,b}\pi(x,y)\lambda(x,y,a,b)\langle E_{x,a}\otimes F_{y,b}\xi,\xi\rangle\\ \omega_{qc}(\mathcal{G})=\sup\sum_{x,y,a,b}\pi(x,y)\lambda(x,y,a,b)\langle E_{x,a}F_{y,b}\xi,\xi\rangle\end{array}\right\}$ maybe difficult to calculate
- Ji, Natarajan, Viddick, Yuen '20: There exists a game with $\omega_q(\mathcal{G})<\omega_{qc}(\mathcal{G})$ refuting Connes embedding problem:if $\mathcal{A}_{XA}=\ell^\infty_A *_1 \ldots *_1 \ell^\infty_A$ ($|X|$ terms) with generators $e_{x,a}=e^*_{x,a}=e^2_{x,a}$ and $\sum_a e_{x,a}=1$, $\forall x\in X$ and

$$t=\sum_{x,y,a,b}\lambda(x,y,a,b)\pi(x,y)e_{x,a}\otimes e_{y,b}\in\mathcal{A}_{XA}\otimes\mathcal{A}_{YB}$$

then $\omega_q(\mathcal{G})=\|t\|_{\mathcal{A}_{XA}\otimes_{\min}\mathcal{A}_{YB}}$, $\omega_{qc}(\mathcal{G})=\|t\|_{\mathcal{A}_{XA}\otimes_{\max}\mathcal{A}_{YB}}$ (Junge et al + Lupini et al)

TOWARDS ANOTHER TENSOR FORMULA

A correlation $p = \{(p(a,b|x,y)_{a,b} : (x,y) \in X \times Y\} \rightsquigarrow \mathcal{N}_p : \mathcal{D}_{XY} \to \mathcal{D}_{AB}$,
$(\mathcal{D}_X \subset M_X$, diagonal matrices, $\mathcal{D}_{XY} = \mathcal{D}_X \otimes \mathcal{D}_Y)$

$$\mathcal{N}_p(\epsilon_{x,x} \otimes \epsilon_{y,y}) = \sum_{a,b \in A} p(a,b|x,y)\epsilon_{a,a} \otimes \epsilon_{b,b}.$$

Incorporating probability distribution π in $\xi_\pi \in \mathbb{C}^{XY} \otimes \mathbb{C}^R$, $R = XY$:

$$\xi_\pi = \sum_{x,y \in X} \sqrt{\pi(x,y)} e_x \otimes e_y \otimes e_x \otimes e_y$$

and the rule function λ in

$$P = \sum_{x,y} \sum_{(a,b) \in \mathcal{E}_{x,y}} \epsilon_{a,a} \otimes \epsilon_{b,b} \otimes \epsilon_{x,x} \otimes \epsilon_{y,y} \in \mathcal{P}_{ABR},$$

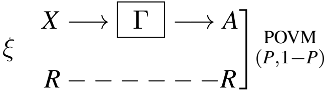
where $\mathcal{E}_{x,y} = \{(a,b) : \lambda(x,y,a,b) = 1\}$, we get

$$\omega_t(\mathcal{G}) = \sup_{p \in \mathcal{C}_t} \mathrm{Tr}((\mathcal{N}_p \otimes \mathrm{id}_R)(\xi_\pi \xi_\pi^*)P).$$

QUANTUM VALUES: ONE PLAYER QUANTUM GAMES

For a collection $\mathcal{Q}$ of quantum channels $\Gamma : M_X \to M_A$, a unit vector $\xi \in \mathbb{C}^X \otimes H_R$, a projection $P \in \mathcal{B}(\mathbb{C}^A \otimes H_R)$ let the $\mathcal{Q}$-value of the pair (ξ, P) be

$$\omega_{\mathcal{Q}}(\xi, P) = \sup_{\Gamma \in \mathcal{Q}} \mathrm{Tr}((\Gamma \otimes \mathrm{id})(\xi\xi^*)P)$$



- $\Gamma : M_X \to M_A$ - quantum channel $\Leftrightarrow \Phi := \Gamma^* : M_A \to M_X$ - u.c.p map
- Adding coeff. get $\Phi : M_A \to M_X \otimes \mathcal{B}(H)$ with Choi matrix $E = (\Phi(\epsilon_{a,a'}))_{a,a'} = (E_{x,x',a,a'})_{x,x',a,a'} \in M_{XA}(\mathcal{B}(H))^+$, a stochastic operator matrix, i.e. $\mathrm{Tr}_A\, E = I_X \otimes I_H$.
- Φ, σ state on $H \rightsquigarrow \Gamma_{E,\sigma} : M_X \to M_A$, $\Gamma_{E,\sigma}(\epsilon_{x,x'}) = \Phi_*(\epsilon_{x,x'} \otimes \sigma) = \sum_{a,a'} \sigma(E_{x,x',a,a'})\epsilon_{a,a'}$. Stinespring's Theorem $\rightsquigarrow$ factorisation $E_{x,x',a,a'} = U^*_{a,x}U_{a',x'}$, where $U = (U_{a,x})_{a,x}$ block operator isometry with $U_{a,x} : H \to K$ s.t. $\Phi(T) = U^*(T \otimes 1_K)U, T \in M_X$.

Collection $\mathcal{Q}$ of channels $\rightsquigarrow$ collection $\mathcal{R}$ of block operator isometries.
$\mathcal{R} \rightsquigarrow QC(\mathcal{R}) = \{\Gamma_{U,\sigma} : U \in \mathcal{R}, \sigma \text{ state}\} \subset \mathcal{Q}$.

Resource space

$U = (U_{a,x})_{a,x}$ block operator isometries $\rightsquigarrow$ universal TRO

$$\mathcal{V}_{X,A} = [u_{a,x} : (u_{a,x})_{a,x} \text{ universal block isometry}]$$

$\rightsquigarrow \mathcal{O}_{X,A} = \mathrm{span}\{u_{a,x} : a \in A, x \in X\} \simeq_{\text{c.i.}} \mathcal{S}_1^{A,X} \equiv_{\text{c.i.}} \mathcal{B}(\mathbb{C}^X, \mathbb{C}^A)^*$, $u_{a,x} \mapsto \epsilon_{x,a} \in \mathcal{S}_1^{A,X}$
$\rightsquigarrow \mathcal{C}_{X,A} = \mathcal{R}_{\mathcal{V}_{X,A}} = [\mathcal{V}_{X,A}^* \mathcal{V}_{X,A}]$, universal C^*-algebra for operator stoch. matrices;
$\rightsquigarrow \mathcal{T}_{X,A} = \mathrm{span}\{e_{x,x',a,a'} := u_{a',x'}^* u_{a,x}, a', a \in A, x, x' \in X\}$ operator system.

Collection $\mathcal{R} = \{(U_{a,x})_{a,x} :\ \text{isometry}\}$ is a resource over (X, A) if

- $\mathcal{R}$ is closed under direct sums;
- $s_{U,\sigma} : \mathcal{T}_{X,A} \to \mathbb{C}$, $\epsilon_{x,x',a,a'} \mapsto \sigma(U_{a,x}^* U_{a',x'})$ is separating

For $u \in M_n(\mathcal{O}_{X,A})$ let $\|u\|_{\mathcal{R}}^{(n)} = \sup_{U \in \mathcal{R}} \|\theta_U^{(n)}(u)\|$, where $\theta_U(u_{a,x}) = U_{a,x}$.
$\rightsquigarrow$ another operator space structure $\mathcal{O}_{X,A}^{\mathcal{R}}$ on $\mathcal{O}_{X,A}$ with id : $\mathcal{O}_{X,A} \mapsto \mathcal{O}_{X,A}^{\mathcal{R}}$ contractive.

Quantum value over a resource

Theorem 1

Let $\mathcal{R}$ be a resource over (X, A), $\xi \in \mathbb{C}^X \otimes H_R$ unit, $P \in \mathcal{B}(\mathbb{C}^A \otimes H_R)$ a projection $P = \sum_n \gamma_n \gamma_n^*$ and $\rho_n = \overline{\mathrm{Tr}_R(\xi \gamma_n^*)} \in \mathcal{S}_1^{A,X} \simeq_{\text{c.i.}} \mathcal{O}_{X,A}$. Then

$$\omega_{\mathcal{R}}(\xi, P) := \sup_{U \in \mathcal{R}, \sigma} \| \mathrm{Tr}((\Gamma_{U,\sigma} \otimes \mathrm{id})(\xi\xi^*)P) \| = \| [\rho_n] \|_{M_{\infty,1}(\mathcal{O}^{\mathcal{R}}_{X,A})}.$$

Proof: If $\xi = \sum_x e_x \otimes \xi_x$ and $\gamma_n = \sum_a e_a \otimes \gamma_{n,a}$ then $\rho_n = \sum_{x,a} \langle \gamma_{n,a}, \xi_x \rangle \epsilon_{x,a}$ and

$$\begin{aligned} &\mathrm{Tr}((\Gamma_{U,\sigma} \otimes \mathrm{id})(\xi\xi^*)P) \\ &= \sum_n \sum_{x,x',a,a'} \sigma(U^*_{a,x} U_{a',x'}) \langle \gamma_{n,a'}, \xi_{x'} \rangle \langle \xi_x, \gamma_{n,a} \rangle \\ &= \sum_n \sigma(\theta_U(\rho_n)^* \theta_U(\rho_n)). \end{aligned}$$

Recall: $\theta_U(\epsilon_{x,a}) = U_{a,x}$.

Quantum no-signalling correlations: classes

- We'd like to play quantum games when Alice and Bob receive quantum states as inputs and apply quantum operations to produce quantum outputs. No-signalling strategies?

Quantum no-signalling (QNS) correlations ($\mathcal{Q}_{\rm ns}$) - quantum channels $\Gamma : M_{XY} \to M_{AB}$:
$\mathrm{Tr}_A\Gamma(\omega_X \otimes \omega_Y) = \mathrm{Tr}_A\Gamma(\omega'_X \otimes \omega_Y)$ and $\mathrm{Tr}_B\Gamma(\omega_X \otimes \omega_Y) = \mathrm{Tr}_B\Gamma(\omega_X \otimes \omega'_Y)$ (Duan-Winter)

A family of classical POVM's:

$\{(E_{x,a})_{a\in A} : x \in X\}$

$\rightsquigarrow E = \sum_{x\in A}\sum_{a\in A} \epsilon_{x,x} \otimes \epsilon_{a,a} \otimes E_{x,a} \in M_{XA}(B(H))^+$

A family of quantum POVM's:

Stochastic operator matrix

$E = (E_{x,x',a,a'}) \in M_{XA}(B(H))^+$ such that $\mathrm{Tr}_A E = I \otimes I_X$.

Classes of QNS $\leftrightarrow$ Choi matrices $(\Gamma(\epsilon_{x,x'} \otimes \epsilon_{y,y'}))_{a,a',b,b'}$:

Local ($\mathcal{Q}_{\rm loc}$)

Convex combinations of $\Phi \otimes \Psi$

Quantum ($\mathcal{Q}_{\rm q}$)

$(\langle E_{x,x',a,a'} \otimes F_{y,y',b,b'}\xi, \xi\rangle)$,
$\xi \in H_A \otimes H_B$, H_A, H_B fin.-dim.

Quantum commuting ($\mathcal{Q}_{\rm qc}$)

$(\langle E_{x,x',a,a'} F_{y,y',b,b'}\xi, \xi\rangle)$, $\xi \in H$

$\phi : M_A \to M_X \otimes \mathcal{B}(H)$,

$\psi : M_B \to M_Y \otimes \mathcal{B}(H)$,

$\Gamma(\rho) = (\phi \cdot \psi)_*(\rho \otimes \omega_{\xi,\xi})$

Quantum non-local games

Quantum graph homomorphism games:

Quantum graphs $\mathcal{U} \subset \mathbb{C}^X \otimes \mathbb{C}^X$, $\mathcal{V} \subset \mathbb{C}^A \otimes \mathbb{C}^A$ are symmetric subspaces, which are skew ($\langle \xi, \sum_{x \in X} e_x \otimes e_x \rangle = 0, \forall \xi \in \mathcal{U}$)
(classical: $\mathcal{U}_G = \text{span}\{e_x \otimes e_y : x \sim_G y\}$).

- Inputs: states ρ in M_{XX}
- Outputs: states σ in M_{AA}
- Strategies: no-signalling $\Gamma : M_{XX} \to M_{AA}$
- Γ is perfect if $\rho \leq P_{\mathcal{U}}$ (ρ is supported in $\mathcal{U}$) $\Rightarrow \Gamma(\rho) \leq P_{\mathcal{V}}$

Question:

Given quantum graphs $\mathcal{U}$, $\mathcal{V}$, is there a perfect strategy in class $\mathcal{Q}_t$ for $\text{Hom}(\mathcal{U}, \mathcal{V})$? If not what is the winning rate?

CLASSES OF QNS AND RESOURCES: THE RESOURCE $\mathcal{R}_{\mathrm{q}}$

- To each QNS class $\mathcal{Q}_{\mathrm{t}}$, we would like to associate a resource $\mathcal{R}_{\mathrm{t}}$ on (XY, AB) of block operator isometries and identify $\omega_{\mathcal{R}_{\mathrm{t}}}(\xi, P)$.

The quantum resource

$$\mathcal{R}_q = \langle U \otimes V : U : \mathbb{C}^X \otimes H_X \to \mathbb{C}^A \otimes K_A, V : \mathbb{C}^Y \otimes H_Y \to \mathbb{C}^B \otimes K_B \rangle$$

(H_X, H_Y finite dim.)

- $QC(\mathcal{R}_q) = \{\Gamma_{W,\sigma} : W \in \mathcal{R}_q, \sigma \text{ state}\} = \mathcal{Q}_q$:

$$\Gamma_{U\otimes V,\omega_{\zeta,\zeta}}(\epsilon_{x,x'} \otimes \epsilon_{y,y'}) = \sum_{a,a',b,b'} \langle (\overbrace{U^*_{a,x} U_{a',x'}}^{E_{x,x',a,a'}} \otimes \overbrace{V^*_{b,y} V_{b',y'}}^{F_{y,y',b,b'}})\zeta, \zeta\rangle \epsilon_{a,a'} \otimes \epsilon_{b,b'}$$

- As $\mathcal{O}_{X,A} \otimes_{\min} \mathcal{O}_{Y,B} \subset \mathcal{V}_{X,A} \otimes_{\min} \mathcal{V}_{Y,B}$, $\mathcal{O}_{X,A} \simeq_{\text{c.i.}} \mathcal{S}_1^{A,X}$, obtain $\mathcal{O}_{XY,AB}^{\mathcal{R}_q} \simeq_{\text{c.i.}} \mathcal{S}_1^{A,X} \otimes_{\min} \mathcal{S}_1^{B,Y}$.
- $\omega_{\mathcal{R}_{\mathrm{q}}}(\xi, P) = \| [\overline{\mathrm{Tr}_R(\xi\gamma_n^*)}] \|^2_{M_{\infty,1}(\mathcal{S}_1^{A,X} \otimes_{\min} \mathcal{S}_1^{B,Y})}$ for $P = \sum_n \gamma_n\gamma_n^*$.

Rank one quantum game G of Cooney-Junge-Palazuelos-Pérez-Garcia '15: $\xi \in \mathbb{C}^{XYR}$ and rank one projection $P = \gamma\gamma^* \in \mathcal{B}(\mathbb{C}^{ABR})$.

Then $\omega_{\mathcal{R}_{\mathrm{q}}}(\xi, P) = \omega^*(G)$.

Classes of QNS and resources: the resource $\mathcal{R}_{qc}$

We say that two families $\mathcal{E} \subset \mathcal{B}(K, L)$ and $\mathcal{F} \subset B(H, K)$ semi-commute if

$$\mathcal{E}^*\mathcal{E} \text{ and } \mathcal{F}\mathcal{F}^* \text{ commute in } \mathcal{B}(K) \quad H \to K \to L$$

The block operator matrices $U = (U_{a,x})_{a,x}$ and $V = (V_{b,y})_{b,y}$ semi-commute if the families $\{U_{a,x}\}_{a,x}$ and $\{V_{b,y}\}_{b,y}$ semi-commute.
The quantum commuting resource

$$\mathcal{R}_{qc} = \langle (U_{a,x}V_{b,y})_{a,b,x,y} : U = (U_{a,x}), V = (V_{b,y}) \text{ semi-comm. isom.} \rangle$$

Theorem 2

We have $QC(\mathcal{R}_{qc}) = \mathcal{Q}_{qc}$.

- For $\mathcal{Q}_{qc} \subset QC(\mathcal{R}_{qc})$ need: $\forall U = (U_{a,x})$, $V = (V_{b,y})$, block isometries with commuting $E_{x,x',a,a'} = U^*_{a,x}U_{a',x'}$ and $F_{y,y',b,b'} = V^*_{b,y}V_{b',y'}$, $\exists\ \tilde{U} = (\tilde{U}_{a,x})$, $\tilde{V} = (\tilde{V}_{b,y})$ block isometries s.t. $\{\tilde{U}_{a,x}\}_{a,x}$ and $\{\tilde{V}_{b,y}\}_{b,y}$ semi-commute and

$$\langle \underbrace{U^*_{a,x}U_{a',x'}}_{E_{x,x',a,a'}} \underbrace{V^*_{b,y}V_{b',y'}}_{F_{y,y',b,b'}} \xi, \xi \rangle = \langle \tilde{V}^*_{b,y}\tilde{U}^*_{a,x}\tilde{U}_{a',x'}\tilde{V}_{b',y'}\xi, \xi \rangle,$$

which is essentially Arveson's lifting theorem: $\psi : M_B \to M_Y \otimes \mathcal{B}(H), \psi(S) = V^*(S \otimes 1_H)V$
$\Rightarrow \exists \rho : \psi(M_B)' \to 1_X \otimes \mathcal{B}(H)$ s.t. $x\psi(S) = V^*\rho(x)(S \otimes 1_H)V, x \in \psi(M_B)'$.

TRO's tensor products and QC-value: $\otimes_{\max}$

A TRO tensor product: TRO's $\mathcal{U}$, $\mathcal{V}$ $\rightsquigarrow$ ternary product on $\mathcal{U} \otimes \mathcal{V}$:
$(u_1 \otimes v_1)(u_2 \otimes v_2)^*(u_3 \otimes v_3) = u_1 u_2^* u_3 \otimes v_1 v_2^* v_3$
$\rightsquigarrow \|w\|_{\max} = \sup\{\|\theta(w)\| : \theta : \mathcal{U} \otimes \mathcal{V} \to \mathcal{B}(H,K)$ ternary morphism$\}$ $\rightsquigarrow \mathcal{U} \otimes_{\max} \mathcal{V}$

Theorem 3

$\theta : \mathcal{U} \otimes \mathcal{V} \to \mathcal{B}(H,L)$ is a ternary morphism iff $\exists$ a Hilbert space K and semi-commuting ternary morphisms $\phi : \mathcal{U} \to \mathcal{B}(K,L)$ and $\psi : \mathcal{V} \to \mathcal{B}(H,K)$ such that $\theta = \phi \cdot \psi$, i.e. $\theta(u \otimes v) = \phi(u)\psi(v)$.

$\rightsquigarrow \|\cdot\|_{\mathcal{R}_{\rm qc}}$: $\|w\|_{\mathcal{R}_{\rm qc}} = \|w\|_{\max}$, $w \in \mathcal{O}_{X,A} \otimes \mathcal{O}_{Y,B} \subset \mathcal{V}_{X,A} \otimes \mathcal{V}_{Y,B}$.

TRO's tensor products and QC-value: $\otimes_{\mathrm{tmax}}$

Say that ternary morphisms $\phi : \mathcal{U} \to \mathcal{B}(H)$, $\psi : \mathcal{V} \to \mathcal{B}(H)$ commute if $\phi(u)\psi(v) = \psi(v)\psi(u)$ and $\phi(u)^*\psi(v) = \psi(v)\phi(u)^*$. For $w \in \mathcal{U} \otimes \mathcal{V}$ set $\|w\|_{\mathrm{tmax}} := \sup\{\|(\phi \cdot \psi)(w)\| : \phi, \psi \text{ commute }\} \rightsquigarrow \mathcal{U} \otimes_{\mathrm{tmax}} \mathcal{V}$ (Kaur-Ruan '02)

Theorem 4

$\|\cdot\|_{\max} = \|\cdot\|_{\mathrm{tmax}}$ on $\mathcal{U} \otimes \mathcal{V}$ and hence $\mathcal{U} \otimes_{\max} \mathcal{V} = \mathcal{U} \otimes_{\mathrm{tmax}} \mathcal{V}$.

Define $\mathcal{S}_1^{A,X} \otimes_{\max} \mathcal{S}_1^{B,Y}$ as an operator subspace of $\mathcal{V}_{X,A} \otimes_{\mathrm{tmax}} \mathcal{V}_{Y,B}$

Corollary

We have $\omega_{\mathcal{R}_{\mathrm{qc}}}(\xi, P) = \|[\rho_n]\|^2_{M_{\infty,1}(\mathcal{S}_1^{A,X} \otimes_{\max} \mathcal{S}_1^{B,Y})}$ for $P = \sum_n \gamma_n \gamma_n^*$, $\rho_n = \overline{\mathrm{Tr}_R(\xi \gamma_n^*)}$.

Let $\mathcal{U} \otimes_\mu \mathcal{V}$ be the symm. Haag. tensor product: $\|M\|_\mu = \inf\{\|u\|_{\mathrm{h}} + \|\sigma(v)\|_{\mathrm{h}} : M = u + v\}$. As for $\|\cdot\|_{\mathrm{h}}$ one takes all c.c. ternary morphisms while in $\|\cdot\|_{\mathrm{tmax}}$ only commuting ones, one gets, using the operator space Grothendieck inequality (Pisier-Shlyahtenko, Haagerup-Musat):

Corollary

We have $\omega_{\mathrm{qc}}(\xi, P) \leq \omega_\mu(\xi, P) := \|[\rho_n]\|^2_{M_{\infty,1}(\mathcal{S}_1^{A,X} \otimes_\mu \mathcal{S}_1^{B,Y})}$. Therefore, $\frac{\omega_{\mathrm{qc}}(\xi,P)}{\omega_{\mathrm{q}}(\xi,P)} \leq 4n$, where $n = \mathrm{rank} P$.

Idea of the proof of Theorem 3

Theorem 3

$\theta : \mathcal{U} \otimes \mathcal{V} \to \mathcal{B}(H, L)$ is a ternary morphism iff $\exists$ a Hilbert space K and semi-commuting ternary morphisms $\phi : \mathcal{U} \to \mathcal{B}(K, L)$ and $\psi : \mathcal{V} \to \mathcal{B}(H, K)$ such that $\theta = \phi \cdot \psi$, i.e. $\theta(u \otimes v) = \phi(u)\psi(v)$.

TRO $\mathcal{U} \rightsquigarrow \mathcal{R}_{\mathcal{U}} = [\mathcal{U}^*\mathcal{U}]$ and $\mathcal{L}_{\mathcal{U}} = [\mathcal{U}\mathcal{U}^*]$.

- Assume $\theta : \mathcal{U} \otimes \mathcal{V} \to \mathcal{B}(H, L)$ is ternary. Then $\pi_\theta^R : \mathcal{R}_{\mathcal{U}} \otimes \mathcal{R}_{\mathcal{V}} \to \mathcal{B}(H)$ is $\pi_\theta^R = \pi_{\mathcal{U}}^R \times \pi_{\mathcal{V}}^R$ for commuting representations $\pi_{\mathcal{U}}^R, \pi_{\mathcal{V}}^R$ of $\mathcal{R}_{\mathcal{U}}, \mathcal{R}_{\mathcal{V}}$.
- Equip $\mathcal{V} \otimes H$ with inner product $\langle v_1 \otimes \xi_1, v_2 \otimes \xi_2 \rangle := \langle \pi_{\mathcal{V}}^R(v_2^* v_1)\xi_1, \xi_2 \rangle$ and get a Hilbert space $\mathcal{V} \otimes_{\mathcal{V}} H$.
- Let $\psi : \mathcal{V} \to \mathcal{B}(H, \mathcal{V} \otimes_{\mathcal{V}} H)$, $\psi(v)\xi = v \otimes_{\mathcal{V}} \xi$. Then $\pi_\psi^R = \pi_{\mathcal{V}}^R$ and $\pi_\psi^L(a)(v \otimes_{\mathcal{V}} \xi) = av \otimes_{\mathcal{V}} \xi$.
- Define $\phi : \mathcal{V} \to \mathcal{B}(\mathcal{V} \otimes_{\mathcal{V}} H, \mathcal{U} \otimes_{\mathcal{U}} (\mathcal{V} \otimes_{\mathcal{V}} H))$, $\phi(a)(v \otimes_{\mathcal{V}} \xi) = a \otimes_{\mathcal{U}} (v \otimes_{\mathcal{V}} \xi)$. We get that ϕ and ψ semi-commute: $H \to \mathcal{V} \otimes_{\mathcal{V}} H \to \mathcal{U} \otimes_{\mathcal{U}} \mathcal{V} \otimes_{\mathcal{V}} H$
- $W : u \otimes_{\mathcal{U}} v \otimes_{\mathcal{V}} \xi \mapsto \theta(u \otimes v)\xi$ is an isometry and $\theta = (W \circ \phi) \cdot \psi$.

Idea of the proof of Theorem 4

Theorem 4

$\|\cdot\|_{\max} = \|\cdot\|_{\mathrm{tmax}}$ on $\mathcal{U} \otimes \mathcal{V}$ and hence $\mathcal{U} \otimes_{\max} \mathcal{V} = \mathcal{U} \otimes_{\mathrm{tmax}} \mathcal{V}$.

Lemma (a la Arveson's commutant lifting)

If $\mathcal{U}$ and $\mathcal{V}$ are TRO's and $\phi : \mathcal{U} \to \mathcal{B}(K, L)$ and $\psi : \mathcal{V} \to \mathcal{B}(H, K)$ semi-commuting ternary morphisms with ψ left non-degenerate, then there exists a $*$-homomorphism $\rho : \mathcal{R}_{\phi(\mathcal{U})} \to (\mathcal{R}_{\psi(\mathcal{V})})'$ s.t. $\rho(b)\psi(v)^* = \psi(v)^* b$, $b \in \mathcal{R}_{\phi(\mathcal{U})}, v \in \mathcal{V}$.

- $\mathcal{U} \rightsquigarrow$ the linking algebra $\mathcal{D}_{\mathcal{U}} = \begin{bmatrix} \mathcal{L}_{\mathcal{U}} & \mathcal{U} \\ \mathcal{U}^* & \mathcal{R}_{\mathcal{U}} \end{bmatrix}$
- Having a c.c.p map $\mathcal{D}_{\mathcal{U}} \to \mathcal{R}_{\mathcal{U}}$, get $\mathcal{R}_{\mathcal{U}} \otimes_{\max} \mathcal{R}_{\mathcal{V}} \subset \mathcal{D}_{\mathcal{U}} \otimes_{\max} \mathcal{D}_{\mathcal{V}}$
- Arveson's lifting applied to semi-commuting $\phi, \psi, \rightsquigarrow \rho : \mathcal{R}_{\phi(\mathcal{U})} \to \mathcal{R}'_{\psi(\mathcal{V})}$ s.t. $\rho \circ \pi^R_\phi$ and π^R_ψ commute
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$$\|(\phi \cdot \psi)(\sum u_i \otimes v_i)\|^2 = \|\sum_{i,j} \psi(v_j)^* \phi(u_j)^* \phi(u_i)\psi(v_i)\| = \|\sum_{i,j} (\rho(\phi(u_j)^* \phi(u_i))\psi(v_j)^* \psi(v_i)\|$$

$$\leq \|\sum_{i,j} u_j^* u_i \otimes v_j^* v_i\|_{\mathcal{R}_{\mathcal{U}} \otimes_{\max} \mathcal{R}_{\mathcal{V}}} = \|\sum_{i,j} u_j^* u_i \otimes v_j^* v_i\|_{\mathcal{D}_{\mathcal{U}} \otimes_{\max} \mathcal{D}_{\mathcal{V}}} = \|\sum_i u_i \otimes v_i\|^2_{\mathcal{D}_{\mathcal{U}} \otimes_{\max} \mathcal{D}_{\mathcal{V}}}$$

i.e. $\|w\|_{\max} \leq \|w\|_{\mathrm{tmax}}$.

THE RESOURCE $\mathcal{R}_{\text{loc}}$ AND LOC-VALUE

$$\mathcal{R}_{\text{loc}} := \{(U \otimes V : U \in \mathcal{B}(\mathbb{C}^X, \mathbb{C}^{AS}), V \in \mathcal{B}(\mathbb{C}^Y, \mathbb{C}^{BT}) \text{ isom. } S, T \text{ fin. sets}\}.$$

$QC(\mathcal{R}_{\text{loc}}) = \mathcal{Q}_{\text{loc}}$:

- Elements of $\mathcal{Q}_{\text{loc}}$ are convex combinations of $\Phi \otimes \Psi$,
- Kraus decomposition $\rightsquigarrow \Phi(\omega) = \sum_{s \in S} U_s \omega U_s^*$, $\Psi(\omega) = \sum_{t \in T} V_t \omega V_t^* \rightsquigarrow U = (U_s)_{s \in S}$, $V = (V_t)_{t \in T}$.

For $\mathcal{X}$, $\mathcal{Y}$ - op. spaces and $\phi : \mathcal{X} \to \mathcal{Y}$, let

$$\|\phi\|_{\text{w,cb}} = \sup\{\|\beta \circ \phi \circ \alpha\|_{\mathcal{S}_2(\ell^2)} : \alpha : R_\infty \to \mathcal{X}, \beta : \mathcal{Y} \to C_\infty \text{ complete contr.}\}$$

$\rightsquigarrow \mathcal{S}_2^{\text{w,cb}}(\mathcal{X}, \mathcal{Y})$ weak cb-Hilbert Schmidt operators and natural op. space structure (Junge-Kubicki-Palazuelos-Pérez-García, '21). For finite dimensional $\mathcal{X}$ and $\mathcal{Y}$ write $\mathcal{X}^* \otimes_{\text{w,cb}} \mathcal{Y} = \mathcal{S}_2^{\text{w,cb}}(\mathcal{X}, \mathcal{Y})$.

Theorem

We have $\mathcal{O}_{XY,AB}^{\mathcal{R}_{\text{loc}}} = \mathcal{S}_1^{A,X} \otimes_{\text{w,cb}} \mathcal{S}_1^{B,Y}$ and $\omega_{\text{loc}}(\xi, P) = \|[\rho_n]\|^2_{M_{\infty,1}(\mathcal{S}_1^{A,X} \otimes_{\text{w,cb}} \mathcal{S}_1^{B,Y})}$

Key: $U = (U_s)_{s \in S} \in \mathcal{B}(\mathbb{C}^X, \mathbb{C}^{AS})$, $V = (V_t)_{t \in T} \in \mathcal{B}(\mathbb{C}^Y, \mathbb{C}^{BT}) \rightsquigarrow (\theta_U \otimes \theta_V)(\omega) = \sum_{s,t} \langle U_s \otimes V_t, \omega \rangle e_s \otimes e_t$, $\alpha(\xi) = \sum_s \langle \xi, e_s \rangle U_s$ and $\beta(\rho) = \sum_t \langle V_t, \rho \rangle e_t$ with $\|(\theta_U \otimes \theta_V)(\omega)\| = \|\beta \circ \omega \circ \alpha\|_2$.

Quantum value of homomorphism game

$\mathcal{U} \in \mathbb{C}^X \otimes \mathbb{C}^X$, $\mathcal{V} \subset \mathbb{C}^A \otimes \mathbb{C}^A$ be quantum graphs; $P_{\mathcal{U}}$ and $P_{\mathcal{V}}$ are the projections onto $\mathcal{U}$ and $\mathcal{V}$ respectively.
Let $\{\xi_i\}_{i\in J}$ be an orthonormal basis of $\mathbb{C}^X \otimes \mathbb{C}^X$ such that $P_{\mathcal{U}} = \sum_{i\in I\subset J} \xi_i\xi_i^*$.
Set $\xi = \sum_{i\in J} \sqrt{\pi(i)}\xi_i \otimes e_i$, where π a public distribution on J, and
$P = \sum_{i\in I} P_{\mathcal{V}} \otimes e_ie_i^* + \sum_{i\notin I} 1 \otimes e_ie_i^* = \sum \eta_{j(i)}\eta_{j(i)}^* \otimes e_ie_i^*$ (predicate).
Let $\rho_{ij} = \sqrt{\pi(i)}\xi_i\eta_{j(i)}^*$ and $M = [\rho_{ij}]_{ij}$, viewed as column vector.

Proposition

TFAE

- $\omega_{\rm qc}(\xi, P) = 1$;
- there is $\Gamma \in \mathcal{Q}_{qc}$ such that for any state $\sigma \in M_{XX}$, we have

$$P_{\mathcal{U}}\sigma P_{\mathcal{U}} = \sigma \Rightarrow P_{\mathcal{V}}\Gamma(\sigma)P_{\mathcal{V}} = \Gamma(\sigma);$$

- $\|M\|_{\max} = 1$.

Alternative game value expressions

- $\mathcal{S}_1^X(M_A)$ - the dual operator space of $M_X \otimes_{\min} \mathcal{S}_1^A \simeq \mathrm{CB}(M_A, M_X)$;
- $\otimes_\varepsilon$ the Banach injective product, $\mathcal{X}, \mathcal{Y}$ finite dim $\rightsquigarrow \mathcal{X}^* \otimes_\varepsilon \mathcal{Y} \simeq \mathcal{B}(\mathcal{X}, \mathcal{Y})$.
- R -finite, $\phi_R : M_R \otimes M_R \to \mathbb{C}$, $a \otimes b \mapsto \mathrm{Tr}(ab)$, $\xi \in \mathbb{C}^{XY} \otimes \mathbb{C}^R$, $P \in P_{ABR}$;
- $\xi\xi^* \otimes P$, as element in $(\mathcal{S}_1^X(M_A) \otimes \mathcal{S}_1^Y(M_B)) \otimes M_R \otimes M_R$, and set $\hat{\mathbb{H}} = \overline{(\mathrm{id} \otimes \phi_R)(\xi\xi^* \otimes P)}$.

Theorem

- $\omega_{\mathrm{loc}}(\xi, P) = \|\hat{\mathbb{H}}\|_{\mathcal{S}_1^X(M_A) \otimes_\varepsilon \mathcal{S}_1^Y(M_B)}$;
- $\omega_{\mathrm{q}}(\xi, P) = \|\hat{\mathbb{H}}\|_{\mathcal{S}_1^X(M_A) \otimes_{\min} \mathcal{S}_1^Y(M_B)}$.

Corollary (Palazuelos-Vidick,'16)

Let $\mathcal{G} = (X, Y, A, B, \lambda, \pi)$ be classical non-local game and $\hat{\mathbb{G}} = (\pi(x,y)\lambda(x,y,a,b))_{x,y,a,b} \in \ell_1^X(\ell_\infty^A) \otimes \ell_1^X(\ell_\infty^B)$. Then

- $\omega_{\mathrm{loc}}(\mathcal{G}) = \|\hat{\mathbb{G}}\|_{\ell_1^X(\ell_\infty^A) \otimes_\varepsilon \ell_1^X(\ell_\infty^B)}$;
- $\omega_{\mathrm{q}}(\mathcal{G}) = \|\hat{\mathbb{G}}\|_{\ell_1^X(\ell_\infty^A) \otimes_{\min} \ell_1^X(\ell_\infty^B)}$.

IDEA OF THE PROOF

- $\mathrm{CB}(M_A, M_X(\mathcal{B}(H))) \simeq \mathcal{S}_1^A \otimes_{\min} M_X \otimes_{\min} \mathcal{B}(H) \simeq \mathrm{CB}(\mathcal{S}_1^X(M_A), \mathcal{B}(H))$;
- $U = (U_{a,x})_{a,x}$ block op. isometry $\rightsquigarrow \Phi_U(\epsilon_{a,a'}) = \sum_{x,x'} \epsilon_{x,x'} \otimes U^*_{a,x} U_{a',x'} \rightsquigarrow$ $T_U : \mathcal{S}_1^X(M_A) \to \mathcal{B}(H)$, $T_U(\epsilon_{x,x'} \otimes \epsilon_{a,a'}) = U^*_{a,x} U_{a',x'}$ c.c.; similar $V = (V_{b,y})_{b,y} \rightsquigarrow S_V : \mathcal{S}_1^Y(M_B) \to \mathcal{B}(K)$
- $\omega_q(\xi, P) = \sup_{(U,V,\sigma)} \overline{\mathrm{Tr}((\Gamma_{U\otimes V,\sigma} \otimes \mathrm{id})(\xi\xi^*)P)} =$ $\sup_{U,V,\sigma} \sigma((T_U \otimes S_V)\underbrace{((\mathrm{id} \otimes \phi_R)(\xi\xi^* \otimes P))}_{\hat{\mathbb{H}}} \rightsquigarrow \omega_{\mathrm{q}}(\xi, P) \le \|\hat{\mathbb{H}}\|_{\mathcal{S}_1^X(M_A)\otimes_{\min}\mathcal{S}_1^Y(M_B)}$

For the reverse inequality:

- using the Paulsen 2-trick, any c.c. map $\Phi : M_A \to M_X \otimes M_n \rightsquigarrow$ to a completely positive $\tilde{\Phi} : M_A \to M_2(M_X \otimes M_n)$ s.t.

$$\tilde{\Phi}(a) = \begin{pmatrix} \phi_1(a) & \Phi(a) \\ \Phi^*(a) & \phi_2(a) \end{pmatrix}, a \in M_A$$

  with ϕ_i u.c.p. Similarly, c.c. $\Psi : M_B \to M_Y \otimes M_n \rightsquigarrow$ u.c.p ψ_i.
- Stinespring $\rightsquigarrow$ block op. isometries $U_i = (U^i_{a,x})_{a,x}$ and $V_i = (V^i_{b,y})_{b,y}$: $\phi_i(\epsilon_{a,a'}) = [(U^i_{a,x})^* U^i_{a',x'}]_{x,x'}$ and $\psi_i(\epsilon_{b,b'}) = [(V^i_{b,y})^* V^i_{b',y'}]_{y,y'} \rightsquigarrow$ if c.c. $T : \mathcal{S}_1^X(M_A) \to M_n$, $S : \mathcal{S}_1^X(M_B) \to M_n$, associated to Φ and Ψ, then $|\langle (T \otimes S)(\hat{\mathbb{H}})\zeta, \eta\rangle| \le \langle (T_{U_1} \otimes S_{V_1})(\hat{\mathbb{H}})\zeta, \zeta\rangle^{1/2} \langle (T_{U_2} \otimes S_{V_2})(\hat{\mathbb{H}})\eta, \eta\rangle^{1/2} \le \omega_{\mathrm{q}}(\xi, P)$.

Thanks for listening!